

On applications of the Maupertuis-Jacobi correspondence for Hamiltonians $F(x, |p|)$ in some 2-D stationary semiclassical problems

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We make use of the Maupertuis – Jacobi correspondence, well known in Classical Mechanics, to simplify 2-D asymptotic formulas based on Maslov’s canonical operator, when constructing Lagrangian manifolds invariant with respect to phase flows for Hamiltonians of the form $F(x, |p|)$. As examples we consider Hamiltonians coming from the Schrödinger equation, the 2-D Dirac equation for graphene and linear water wave theory.

1. INTRODUCTION

Maupertuis – Jacobi correspondence [1–3] allows to relate two Hamiltonians $\mathcal{H}(x, p, E)$ and $H(x, p, E)$ having in common a regular energy surface Σ ; it preserves the integral curves on Σ up to a reparametrization of time. As it was shown in [5, 6] this principle is also useful in determining the semiclassical spectral asymptotics for a selfadjoint h -pseudodifferential operator $\mathcal{H}(x, hD_x, E; h)$ having $\mathcal{H}(x, p, E)$ as principal semi-classical symbol. The other Hamiltonian $H(x, p, E)$ is assumed to enjoy nice properties, such as local integrability near Σ . Then we can construct some compact Lagrangian manifolds invariant by the flow of

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$\mathcal{H}(x, p, E)$, and then determine quasimodes for $\mathcal{H}(x, hD_x, E; h)$ microlocalized in a neighborhood of Σ .

In this communication we want to show that Maupertuis – Jacobi correspondence allows to construct non compact Lagrangian manifolds appearing in the scattering problem for $\mathcal{H}(x, hD_x, E; h)$, or the problem about Green function asymptotics. Here it is assumed that $H(x, p, E)$ is a Finsler symbol, which implies the existence of special coordinates near the singular part of the Lagrangian manifold. When combining Maupertuis – Jacobi correspondence with new formulas for Maslov canonical operator [8], we show that the corresponding asymptotics can be presented in a rather explicit and simple form. We restrict ourselves to 2-D case and apply our considerations to examples from the Schrödinger (or Helmholtz) equation, the two-dimensional Dirac equation for graphene and Pseudodifferential operators from the linear water wave theory.

2. LAGRANGIAN MANIFOLDS INVARIANT WITH RESPECT TO HAMILTONIANS $F(X, |P|)$, THE EIKONAL COORDINATES AND MAUPERTUIS – JACOBI CORRESPONDENCE

Let $F(x, z)$, $x \in \mathbb{R}^2$, $z \in [0, \infty)$ be a smooth function, and E a real parameter. Assume that the equation $F(x, z) = E$ has the unique solution $z = 1/C(x, E)$, where $C(x, E)$ is a smooth positive bounded function, such that $C(x, E) \geq c_0(E) > 0$. Also we assume that $|\frac{\partial F}{\partial z}(x, \frac{1}{C(x, E)})| \geq c_1(E) > 0$, where $c_0(E), c_1(E)$ are some positive constants. Consider the Hamiltonians $\mathcal{H}(x, p, E) = F(x, |p|) - E$, $H(x, p, E) = C(x, E)|p| - 1$ in the phase space $\mathbb{R}_{p,x}^4 = T^*\mathbb{R}^2$ together with the Hamiltonian systems

$$(a) \quad \frac{dp}{dt} = -\mathcal{H}_x, \quad \frac{dx}{dt} = \mathcal{H}_p; \quad (b) \quad \frac{dp}{d\tau} = -H_x, \quad \frac{dx}{d\tau} = H_p \quad (1)$$

We recall that $C(x, E)|p|$ defines a (reversible) Finsler symbol on $T^*\mathbb{R}^2$ [4, 7].

Let $\mathcal{Q} = \mathbb{R}$ or $\mathcal{Q} = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{Q} \rightarrow T^*\mathbb{R}^2$, $\varphi \mapsto (P^0(\phi, E), X^0(\phi, E))$ be a smooth embedding with image Λ^1 such that

$$\mathcal{H}(X^0(\phi, E), P^0(\phi, E), E) = 0 \quad \text{and} \quad |P^0(\phi, E)|C(X^0(\phi, E)) = 1$$

Consider the solutions $(\mathcal{P}(t, \phi, E), \mathcal{X}(t, \phi, E))$ and $(P(\tau, \phi, E), X(\tau, \phi, E))$ to systems (1) (a) and (b) respectively with initial data on Λ^1 . Due to general properties of Hamiltonian

systems, we have

$$\mathcal{H}(\mathcal{X}(t, \phi, E), \mathcal{P}(t, \phi, E), E) = 0 \quad \text{and} \quad |P(\tau, \phi, E)| C(X(\tau, \phi, E)) = 1$$

and because of Maupertuis – Jacobi correspondence, trajectories $(\mathcal{P}(t, \phi, E), \mathcal{X}(t, \phi, E))$ and $(P(\tau, \phi, E), X(\tau, \phi, E))$ coincide modulo a reparametrization of time. Indeed one has

$$\begin{aligned} \frac{d\mathcal{P}}{dt} &= -\mathcal{H}_x(\mathcal{P}, \mathcal{X}) = -R(\mathcal{X})H_x(\mathcal{P}, \mathcal{X}) = R(\mathcal{X})\frac{dP}{d\tau} \\ \frac{d\mathcal{X}}{dt} &= \mathcal{H}_p(\mathcal{P}, \mathcal{X}) = R(\mathcal{X})H_p(\mathcal{P}, \mathcal{X}) = R(\mathcal{X})\frac{dX}{d\tau} \end{aligned} \quad (2)$$

where

$$R(x) = \lim_{z \rightarrow 1/C(x, E)} \frac{F(x, z) - E}{zC(x, E) - 1} = z \frac{\partial F}{\partial z}(x, z) \Big|_{z=1/C(x, E)}$$

Changing time t by time $\tau = \tau(t, \phi, E)$, by using the equation

$$\frac{d\tau}{dt} = R(\mathcal{X}(t, \phi, E)), \quad \tau|_{t=0} = 0, \quad (3)$$

we get (inverting the equation $\tau = \tau(t, \phi, E)$ we obtain $t = t(\tau, \phi, E)$)

$$\begin{aligned} (\mathcal{P}(t, \phi, E), \mathcal{X}(t, \phi, E)) &= (P(\tau, \phi, E), X(\tau, \phi, E)) \Big|_{\tau=\tau(t, \phi, E)} \iff \\ (P(\tau, \phi, E), X(\tau, \phi, E)) &= (\mathcal{P}(t, \phi, E), \mathcal{X}(t, \phi, E)) \Big|_{t=t(\tau, \phi, E)}. \end{aligned} \quad (4)$$

In $T^*\mathbb{R}^2$ the solutions $(P(\tau, \phi, E), X(\tau, \phi, E))$ and $(\mathcal{P}(t, \phi, E), \mathcal{X}(t, \phi, E))$ define the phase flows (that we assume to be defined for all time)

$$\begin{aligned} \Lambda^2 &= \bigcup_{t \in \mathbb{R}} g_{\mathcal{H}}^t \Lambda^1 = \{(p, x) = (\mathcal{P}(t, \phi, E), \mathcal{X}(t, \phi, E)), \phi \in Q, t \in \mathbb{R}\} = \\ &\bigcup_{t \in \mathbb{R}} g_H^t \Lambda^1 = \{(p, x) = (P(\tau, \phi, E), X(\tau, \phi, E)), \phi \in Q, \tau \in \mathbb{R}\} \end{aligned} \quad (5)$$

In particular Λ^2 is invariant under $g_{\mathcal{H}}^t$ and g_H^t . Note that we could replace \mathbb{R}^2 by an open domain $\Omega \subset \mathbb{R}^2$ and consider instead the maximal classical trajectories $g_H^t(\rho)$, $\rho \in \Lambda^1$ and $t \in (T_-(\rho), T_+(\rho))$, and similarly for $g_{\mathcal{H}}^t$. The parameters t and τ are called *proper times*. Once Λ^2 is a smooth manifold, it becomes an embedded Lagrangian manifold, and (t, ϕ) and (τ, ϕ) are just two different coordinate systems on Λ^2 . It is convenient to relate objects belonging to either Hamiltonians, such as eikonals or half-densities. In particular:

Lemma 1. The following properties hold:

1) The Jacobians of the transformation $(t, \phi) \mapsto (\tau, \phi)$ or its inverse verify

$$\det \frac{\partial(\tau(t, \phi, E), \phi)}{\partial(t, \phi)} = \frac{d\tau}{dt} = R(\mathcal{X}(t, \phi, E)), \quad (6)$$

$$\det \frac{\partial(t(\tau, \phi, E), \phi)}{\partial(\tau, \phi)} = \frac{dt}{d\tau} = 1/R(X(\tau, \phi, E)) \quad (7)$$

and the Jacobians $J = \det \frac{\partial X}{\partial(\tau, \phi)}$ (resp. $\mathcal{J} = \det \frac{\partial \mathcal{X}}{\partial(t, \phi)}$) in coordinates (τ, ϕ) (resp. (t, ϕ)) are related by

$$\mathcal{J}(t, \phi) = R(\mathcal{X}(\tau, \phi, E))J(\tau, \phi) \Big|_{\tau=\tau(t)}. \quad (8)$$

2) The action function (eikonal) on Λ^2 is

$$\begin{aligned} s(t, \phi) &\equiv \int_{(0,0)}^{(t,\phi)} \mathcal{P}(t, \phi, E) d\mathcal{X}(t, \phi, E) = s_0(\phi) + \tau \\ s_0(\phi) &= \int_0^\phi P^0(\phi, E) dX^0(\phi, E) \end{aligned} \quad (9)$$

3) The Jacobians J, \mathcal{J} satisfy to the relations:

$$|J| = C(X(\tau, \phi))|X_\phi|, \quad |\mathcal{J}| = R(\mathcal{X}(t, \phi))C(\mathcal{X}(t, \phi))|X_\phi|. \quad (10)$$

Proof. The first equalities (6), (7), (8) hold since $\frac{\partial \phi}{\partial \tau} = \frac{\partial \phi}{\partial t} = 0$. The proof of (9) follows from chain of equalities

$$\begin{aligned} s(t, \phi) &= \int_{(0,0)}^{(t,\phi)} \mathcal{P}(t', \phi, E) d\mathcal{X}(t', \phi, E) = \int_{(0,0)}^{(\tau,\phi)} P(\tau', \phi, E) dX(\tau', \phi, E) = \\ &= \int_0^\phi P^0(\phi, E) dX^0(\phi, E) + \int_0^\tau P(\tau', \phi, E) \frac{dX}{d\tau'}(\tau', \phi, E) d\tau' = \\ &= \int_0^\phi P^0(\phi, E) dX^0(\phi, E) + \int_0^\tau |P(\tau', \phi, E)| C(X(\tau', \phi, E), E) d\tau' = s_0(\phi) + \tau \end{aligned}$$

The assertion 3) is proved in [8, 11], using that $H(x, p, E)$ is a Finsler symbol. \square

The pair $(\tilde{\tau} = s_0(\phi) + \tau, \phi)$ are called *eikonal coordinates* on Λ^2 (see [8]). There are two important examples of curves Λ^1 in applications: $\Lambda_s^1 = \{p_1 = 0, p_2 = k, x_1 = \phi, x_2 = a, \phi \in \mathbb{R}\}$ which appears in scattering problems and $\Lambda_G^1 = \{p_1 = b \cos \phi, p_2 = b \sin \phi, x_1 = a_1, x_2 = a_2, \phi \in \mathbb{R}/2\pi\mathbb{Z}\}$ which appears in problems about the Green functions. Easy to check that in these cases $s_0(\phi) = 0$ and $\tilde{\tau} = \tau$.

3. RELATIONSHIP WITH MASLOV CANONICAL OPERATOR

We endow the Lagrangian manifold Λ^2 with the measure $d\mu = dt \wedge d\phi$; let $A(t, \phi)$ be a smooth function on Λ^2 and

$$\psi = K_{\Lambda^2}^h A(t, \phi). \quad (11)$$

where $K_{\Lambda^2}^h$ is Maslov canonical operator. We want to pass in $K_{\Lambda^2}^h A(t, \phi)$ from coordinates (t, ϕ) to eikonal-coordinates (τ, ϕ) preserving the measure $d\mu$.

Theorem. The following equalities hold:

$$\begin{aligned} \psi &= K_{\Lambda^2}^h \left[A(t(\tau, \phi), \phi) / \sqrt{\det \frac{\partial(\tau, \phi)}{\partial(t, \phi)}} \right] = \\ K_{\Lambda^2}^h \left[\frac{A(t(\tau, \phi), \phi)}{\sqrt{R(X(\tau, \phi))}} \right] &= \frac{1}{\sqrt{R(x)}} K_{\Lambda^2}^h \left[A(t(\tau, \phi), \phi) \right] (1 + \mathcal{O}(h)). \end{aligned} \quad (12)$$

Proof. It follows easily from (6) and the commutation formula between the Pseudodifferential operator $\hat{Q} = Q(x, hD_x)$ and Maslov canonical operator [9, 10]: $\hat{Q} K_{\Lambda^2}^h [A(t, \phi)] = K_{\Lambda^2}^h [Q(x, p)|_{\Lambda^2} A(t, \phi)] (1 + \mathcal{O}(h))$. \square

Recall that the canonical operator has different representations in the neighborhood of regular points (where $J = \det \frac{\partial X}{\partial(\tau, \phi)} \neq 0$) and in the neighborhood of singular (focal) points (where $J = \det \frac{\partial X}{\partial(\tau, \phi)} = 0$). According to (10) the point $(P(\tau, \phi), X(\tau, \phi)) \in \Lambda^2$ is singular (focal) if $X_\phi(\tau, \phi) = 0$. It was proved in [8] that under existence of the eikonal coordinates, $\det(P, P_\phi) \neq 0$ in the neighborhood of the focal points. Here (P, P_ϕ) is the 2×2 matrix constructed from vector columns P and P_ϕ . Thus the Lagrangian manifold could be covered by regular charts Ω_j^{reg} with $X_\phi(\tau, \phi) \neq 0$ and singular charts Ω_j^{sing} with $\det(P, P_\phi) \neq 0$. Let $\{\mathbf{e}_j(\tau, \phi)\}$ be a (finite) partition of unity subordinated to the charts $\Omega_j^{\text{reg}}, \Omega_j^{\text{sing}}$. Then due to Lemma 1 the contribution of a regular chart to the canonical operator is

$$\psi_j = \frac{e^{-i\frac{\pi}{2}\mathbf{m}_j}}{\sqrt{R(x)C(x)|X_\phi|}} e^{i\frac{\tau}{h}} A(\tau, \phi) \mathbf{e}_j(\tau, \phi) \Big|_{(\tau, \phi)=(\tau_j(x), \phi_j(x))} \quad (13)$$

where $(\tau_j(x), \phi_j(x))$ is the solution to the (vector) equation $X(\tau, \phi) = x$ in the chart Ω_j^{reg} and \mathbf{m}_j is the Maslov index of Ω_j^{reg} (see below). The contribution of a singular chart is [8]

$$\psi_j = \frac{e^{-i\frac{\pi}{2}\mathbf{m}_j^s} e^{i\frac{\pi}{4}}}{\sqrt{hR(x)}} \int_{\mathbb{R}} e^{i\frac{\tau}{h}} \sqrt{|\det(P, P_\phi)|} A(\tau, \phi) \mathbf{e}_j(\tau, \phi) \Big|_{\tau=\tau_j(x, \phi)} d\phi \quad (14)$$

where $\tau_j(x, \phi)$ is the solution to the scalar equation $\langle P(\tau, \phi), x - X(\tau, \phi) \rangle = 0$ in the chart Ω_j^{sing} and \mathbf{m}_j^s is the Maslov index of Ω_j^{sing} .

According to [8] Maslov index \mathbf{m}_j coincides with Morse index of the trajectory starting from the point (P, X) with coordinates $(\tau = 0^+, \phi)$ and coming to the point $(P(\tau, \phi), X(\tau, \phi)) \in \Omega_j^{\text{reg}}$: it equals to a number of zeroes of Jacobian $J = \det \frac{\partial X}{\partial(\tau', \phi)}$ (or the function $X_\phi(\tau', \phi)$) when τ' runs from 0^+ to τ . To find the index \mathbf{m}_j^s of a singular chart Ω_j^{sing} one need to take an arbitrary regular point $(P(\tau, \phi), X(\tau, \phi)) \in \Omega_j^{\text{sing}}$ and compare the signs of $J = \det \frac{\partial X}{\partial(\tau', \phi)}$ and $\det(P, P_\phi)$. Then \mathbf{m}_j^s equals Morse index of $(P(\tau, \phi), X(\tau, \phi))$ if they coincide, and Morse index plus 1 otherwise. Finally to construct the canonical operator one should patch all ψ_j together (see [9, 10]). At last note that integral (14) could be expressed in the form of Airy or Pearcey functions (see explicit formulas in [8]) under the assumption that the certain subset of Lagrangian singularities $\{(P(\tau, \phi), X(\tau, \phi))|_{X_\phi=0}\}$ are in the so-called general position ([1, 9]).

We consider the following example. The Lagrangian manifold presented in Fig. 1 has 2 caustics (red lines). Under the area in configuration space between edges of a caustic the Lagrangian manifold is folded into 3 leaves. So in this area 3 functions of the form (13) are to be patched together. Under the area “outside caustics” there is only one leave of the manifold, equation $X(\tau, \phi) = x$ has a unique solution and the canonical operator takes the form of (13) with a single function. In the vicinity of caustic edges canonical operator is a sum of a regular (13) and singular (14) parts.

4. EXAMPLES

Let us present several examples of application of the Maupertuis – Jacobi correspondence for the construction of Maslov canonical operator. We do not discuss here further applications to Partial Differential Equations.

Example 1 (from the Schrödinger equation, see [12, 13]). Let $U(x)$ be a smooth bounded function, $U(x) < E$. Consider the classical Hamiltonian $\mathcal{H}(x, p) = F(x, |p|) = \frac{p^2}{2} + U(x)$. Then

$$C(x, E) = \frac{1}{\sqrt{2(E - U(x))}}, \quad R(x) = z^2 \Big|_{z=1/C(x)} = 2(E - U(x)) \quad (15)$$

and

$$\psi(x) = \frac{1}{\sqrt{2(E - U(x))}} K_{\Lambda^2}^h \left[A(t(\tau, \phi), \phi) \right]. \quad (16)$$

Example 2 (from the two-dimensional Dirac equation for graphene, [14]). Let $U(x), m(x)$ be smooth bounded functions. Consider the effective Hamiltonians $\mathcal{H}^\pm(x, p) = F(x, |p|) =$

$U(x) \pm \sqrt{p^2 + m(x)^2}$. Then

$$C(x, E) = \frac{1}{\sqrt{(E - U)^2 - m^2}}, \quad R = \pm \frac{z^2}{\sqrt{z^2 + m(x)^2}} \Big|_{z=1/C} = \frac{(E - U(x))^2 - m^2(x)}{E - U(x)}, \quad (17)$$

and

$$\psi = \frac{\sqrt{E - U(x)}}{\sqrt{(E - U(x))^2 - m(x)^2}} K_{\Lambda^2}^h \left[A(t(\tau, \phi), \phi) \right]. \quad (18)$$

Example 3 (from the water waves theory, [6, 15, 16]). Let $D(x) > 0$ be the smooth function, representing the depth of a basin, and consider the effective Hamiltonian $\mathcal{H}(x, p) = F(x, |p|) = \sqrt{|p| \tanh(|p|D(x))} - E$. It is easy to see that there exists a unique smooth positive solution $y = Y(\mathcal{E}(x))$ to the equation $\sqrt{y \tanh(y)} = \mathcal{E}(x) = \sqrt{D(x)E}$ and

$$\begin{aligned} C(x, E) &= \frac{D(x)}{y(E\sqrt{D(x)})}, \quad R = z \frac{D(x)z / \cosh^2(zD(x)) + \tanh(zD(x))}{2\sqrt{z \tanh(zD(x))}} \Big|_{z=\frac{1}{C}} = \\ &= \frac{(y^2 - y^2 \tanh^2(y)) + y \tanh(y)}{2\sqrt{D} \sqrt{y \tanh(y)}} \Big|_{y=Y(\sqrt{D(x)E})} = \frac{y^2 - D(x)^2 E^4 + D(x) E^2}{2D(x)E} \Big|_{y=Y(\sqrt{D(x)E})} \end{aligned} \quad (19)$$

The Hamiltonian system with the Hamiltonian $H(x, p, E) = C(x, E)|p|$ has the form

$$\frac{dp}{d\tau} = -|p| \frac{\partial}{\partial x} \left(\frac{D(x)}{Y(E\sqrt{D(x)})} \right), \quad \frac{dx}{d\tau} = \frac{p}{|p|} \frac{D(x)}{Y(E\sqrt{D(x)})}. \quad (20)$$

It contains function $Y(\mathcal{E}(x))$ and its derivative $Y'(\mathcal{E}(x))$ which makes difficult to finding its inverse. Let us show how to rewrite this system in a form without the function $Y(\mathcal{E}(x))$.

Along the trajectories (P, X) the following equalities hold:

$$H(X, P, E) = 0 \quad \Leftrightarrow \quad C(X, E)|P| = 1 \quad \Rightarrow \quad Y(E\sqrt{D(X)}) = \frac{D(X)}{C(X, E)} = D(X)|P|. \quad (21)$$

Then differentiating the equation for $Y(\mathcal{E}(x))$ we have

$$Y'(\tanh Y + Y(1 - \tanh^2 Y)) = 2\sqrt{Y \tanh Y}. \quad (22)$$

This gives for the solution (P, X)

$$Y'(E\sqrt{D(X)}) = \frac{2Y\mathcal{E}}{Y^2 + \mathcal{E}^2 - \mathcal{E}^4} \Big|_{(P, X)} = \frac{2|P|\sqrt{D(X)}E}{D(X)|P|^2 + E^2 - D(X)E^4}. \quad (23)$$

Inserting these equalities (21) and (23) into the Hamiltonian system we finally have

$$\frac{dp}{d\tau} = -\frac{p^2 - E^4}{D(x)p^2 + E^2 - D(x)E^4} \cdot \frac{\partial D(x)}{\partial x}, \quad \frac{dx}{d\tau} = \frac{p}{p^2} \quad (24)$$

To write out the canonical operator we also insert $Y|_{(P,X)}$ into the expression (19) for R :

$$R|_{(P,X)} \equiv R(X, P, E) = \frac{D(X)P^2 - D(X)E^4 + E^2}{2E} \quad (25)$$

Taking into account the last expression for R and that $C(X) = 1/|P|$ we obtain the formula (13) for canonical operator in a regular point in this case

$$\psi_j(x) = \frac{e^{i\frac{\tau}{h}} e^{-i\frac{\pi}{2}\mathbf{m}_j}}{\sqrt{|X_\phi(\tau, \phi)|}} \sqrt{\frac{2E |P(\tau, \phi)|}{D(X(\tau, \phi))P^2 - D(X(\tau, \phi))E^4 + E^2}} A(\tau, \phi) \mathbf{e}_j(\tau, \phi) \Big|_{(\tau, \phi) = (\tau_j(x), \phi_j(x))} \quad (26)$$

As R depends on p , it is not convenient to factor out $1/\sqrt{R(x, hD_x, E)}$ from the canonical operator. Near the focal point we write instead

$$\psi_j(x) = e^{-i\frac{\pi}{2}\mathbf{m}_j^s} e^{i\frac{\pi}{4}} \frac{\sqrt{2E}}{\sqrt{h}} \int_{\mathbb{R}} \frac{\sqrt{|\det(P(\tau, \phi), P_\phi(\tau, \phi))|} e^{i\frac{\tau}{h}} A(\tau, \phi) \mathbf{e}_j(\tau, \phi)}{\sqrt{D(X(\tau, \phi))P^2(\tau, \phi) - D(X(\tau, \phi))E^4 + E^2}} \Big|_{\tau=\tau_j(x, \phi)} d\phi \quad (27)$$

Example 4 (from the water waves theory with surface tension, [6, 15, 16]). We modify Hamiltonian in Example 3 according to $\mathcal{H}(x, p) = F(x, |p|) - E = \sqrt{|p| \tanh(|p|D(x))(1 + \mu(x)|p|^2)} - E$, $x \in \mathbb{R}^2$, where $\mu(x) > 0$ is a smooth function, representing the surface tension of the fluid. Let also $\nu(x) = E(\mu(x))^{1/4}$, $\mathcal{E}(x) = E(D(x))^{1/2}$. The relation $\mathcal{H}(x, p) = 0$ can be rewritten in a functional form as $f(y, \mathcal{E}, \nu) = 0$, where $f(y, \mathcal{E}, \nu) = y \tanh(y) - \mathcal{E}^2(1 + y^2 \frac{\nu^4}{\mathcal{E}^4})^{-1}$ is smooth on \mathbb{R}_+^3 , and because $\frac{\partial f}{\partial y}(y, \mathcal{E}, \nu) > 0$, the implicit functions theorem gives $y = Y(\mathcal{E}, \nu)$ where Y is smooth in $(\mathcal{E}, \nu) \in \mathbf{R}_+^2$. Since μ and D are smooth functions, it follows also from the implicit function theorem in Fréchet space $C^\infty(\mathbb{R}^2)$ that $y = Y(\mathcal{E}, \nu) \in C^\infty(\mathbb{R}^2)$. As above, the equations of motion with Hamiltonian $C(x, E)|p|$ have the form

$$\frac{dp}{d\tau} = -|p| \frac{\partial}{\partial x} \left(\frac{D(x)}{Y(E\sqrt{D(x)}, E\mu(x)^{1/4})} \right), \quad \frac{dx}{d\tau} = \frac{p}{|p|} \frac{D(x)}{Y(E\sqrt{D(x)}, E\mu(x)^{1/4})}. \quad (28)$$

As above they simplify on $H(x, p, E) = C(x, E)|p| - 1 = 0$ to

$$\frac{dp}{d\tau} = -|p| \left(\frac{1}{y} \frac{\partial D}{\partial x} - \frac{D(x)}{y^2} \frac{dY}{dx} \right) \Big|_{y=D(x)|p|}, \quad \frac{dx}{d\tau} = \frac{p}{p^2} \quad (29)$$

Here dY/dx is found by differentiating the equation $y = Y(\mathcal{E}(x), \nu(x))$:

$$\begin{aligned} \frac{dY}{dx} &= -\left(\frac{\partial f}{\partial y}\right)^{-1} \left(\frac{\partial f}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial x} + \frac{\partial f}{\partial \nu} \frac{\partial \nu}{\partial x} \right), \\ \frac{\partial f}{\partial y} &= \tanh(y) + y(1 - \tanh^2(y)) + 2y\mathcal{E}^{-2}\nu^4(1 + y^2 \frac{\nu^4}{\mathcal{E}^4})^{-2} > 0. \end{aligned}$$

Substituting this derivatives into (29) with $y(E\sqrt{D(X)}) = |P|D(X)$ we get a system similar to (24), and can we obtain also an expression for R as in (25). So we are able to get a representation of Maslov canonical operator as in (26) and (27).

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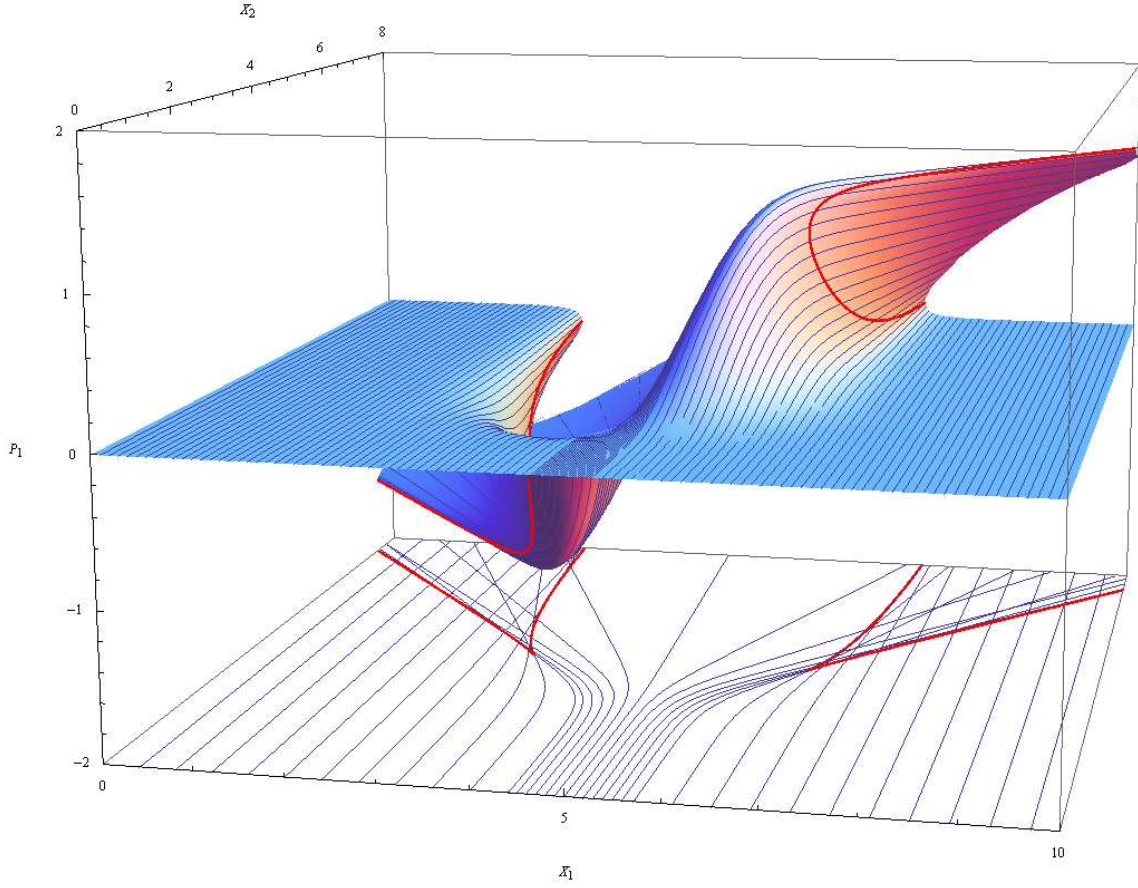


Figure 1. Lagrangian manifold, characteristics (blue lines) and caustic (red lines): in the phase space (in coordinates $(x_1, x_2, p_1) \in \mathbb{R}_{x,p}^4$) and in projection to the configuration space \mathbb{R}_x^2 . The

Lagrangian manifold $\Lambda^2 = \bigcup g_H^t \Lambda^1$ corresponds to a scattering problem with initial curve

$\Lambda^1 = \{p = (0, 2), x = (\phi, 0), \phi \in \mathbb{R}\}$, the Hamiltonian is $H(x, p) = |p|/(E - U(x))$, where

$E = 2, U(x) = \mathbf{e}(x)e^{-(x_1-5)^2-(x_2-3)^2}$ and $\mathbf{e}(x)$ – is a cut-off function

$$\mathbf{e}(x) = 0, x_2 \leq 0, \mathbf{e}(x) = 1, x_2 \geq 1.$$